Symmetry Analysis of Boundary Layer Equations of an Upper Convected Maxwell Fluid with Magnetohydrodynamic Flow

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Steady state boundary layer equations of an upper convected Maxwell fluid with magnetohydrodynamic (MHD) flow are considered. The strength of the magnetic field is assumed to be variable with respect to the location. Using Lie group theory, group classification of the equations with respect to the variable magnetic field is performed. General boundary conditions including stretching sheet and injection are taken. Restrictions imposed by the boundary conditions on the symmetries are discussed. Special functional forms of boundary conditions for which similarity solutions may exist are derived. Using the symmetries, similarity solutions are presented for the case of constant strength magnetic field. Stretching sheet solutions with or without injection are presented. Effects of physical parameters on the solutions are depicted.

Key words: Upper Convected Maxwell Fluid; Boundary Layers; Stretching Sheet; Magnetohydrodynamic (MHD) Flow; Lie Group Theory; Group Classification.

1. Introduction

A wide range of fluids exhibits complex behaviour which can not be examined within the context of the Newtonian fluid theory which predicts a linear relationship between the shear stress and the velocity gradient. Usually, the stress constitutive relations inherit complexities which lead to highly nonlinear equations of motion with many terms. To simplify the extremely complex equations with excess terms, one alternative is to use the boundary layer assumptions which are known to effectively reduce the complexity of Navier-Stokes equations and reduce drastically the computational time. Since there are many non-Newtonian models and further new models are proposed continuously, the boundary layer theory for each proposed model also appears in the literature. It is beyond the scope of this work to review vast literature on the boundary layers of non-Newtonian fluids. A limited work on the topic can be referred as examples [1 – 20].

In this work, the boundary layer equations of upper convected Maxwell fluids are considered. The steady-state case with two-dimensional flow is considered. A variable magnetic field with location is introduced into the equations. Some of the closely related recent works are the following: Sadeghy et al. [21] examined the Sakiadis flow of a Maxwell fluid. Hayat et al. [22] considered MHD flow of an upper convected Maxwell fluid on a porous stretching plate. Abbas et al. [23] examined a similar problem with porous channel flow. Hayat and Abbas [24] further investigated the influence of chemical reactions on the flow of a Maxwell fluid through a porous channel. Sadeghy et al. [25] examined the stagnation point flow of a Maxwellian fluid. MHD flow over impulsively stretching sheet was investigated by Pahlavan and Sadeghy [26]. Thermal effects were further considered by Aliakbar et al. [27].

In all mentioned works [21 – 27], the partial differential equations were converted to ordinary differential equations via a similarity transformation found by ad hoc methods. The resulting ordinary differential equations were then solved by some series type solutions, and the homotopy analysis method (HAM) has been employed frequently. A detailed investigation of the equations and the complete symmetries of the equations using Lie group theory [28, 29] are lacking in the literature. A group classification with respect to the variable magnetic field is also new in this study. Since fairly general boundary conditions are taken, moving plate, stretching sheet or porous plate with injection cases can be covered. It is well known that boundary conditions impose severe restrictions on the symmetries of differential equations which may lead to even annihilation of all symmetries. In our problem, the restrictions imposed by the boundary conditions are treated in a fairly general manner, and it is found that some symmetries are stable while some others are unstable after application of the boundary conditions. The treatment of boundary conditions with respect to the symmetries of boundary layers of an upper-convected Maxwell fluid is new and may guide researchers looking for similarity transformations of such boundary value problems. Finally, a similarity transformation stemming from a scaling symmetry is employed to transfer the partial differential system to an ordinary differential system for the case of constant magnetic strength. Stretching sheet solutions with or without injection are presented by numerically solving the resulting ordinary differential equations. Effects of elasticity number, magnetic number, inverse Reynolds number, and injection velocities on the solutions are depicted in the figures.

2. Equations of Motion

The two-dimensional boundary layer equations for a steady state upper convected Maxwell fluid with MHD flow is [26]

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0, \tag{1}$$

$$u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}+\lambda\left[u^{*2}\left(\frac{\partial^2 u^*}{\partial x^{*2}}\right)+v^{*2}\left(\frac{\partial^2 u^*}{\partial y^{*2}}\right)+2u^*v^*\left(\frac{\partial^2 u^*}{\partial x^*\partial y^*}\right)\right]=\frac{\mu}{\rho}\left(\frac{\partial^2 u^*}{\partial y^{*2}}\right)-\frac{\sigma B_0^2(x^*)}{\rho}u^*, \tag{2}$$

where x^* is the spatial coordinate along the plate, y^* is vertical to it, u^* and v^* are the velocity components in x^* and y^*-coordinates. λ is the relaxation time of the fluid, μ the viscosity, ρ the density, σ the electrical conductivity, and $B_0(x^*)$ the variable magnetic field strength.

The associated boundary conditions are taken as fairly general

$$u^*(x^*,0)=U^*(x^*),\quad v^*(x^*,0)=V^*(x^*),\quad u^*(x^*,\infty)=0, \tag{3}$$

so that special forms of the functions $U^*(x^*)$ and $V^*(x^*)$ which do not annihilate symmetries and, hence, do not spoil similarity transformations can be determined in a systematic manner. By introducing the dimensionless variables for universality of the results

$$x=\frac{x^*}{L},\quad y=\frac{y^*}{L},\quad u=\frac{u^*}{U},\quad v=\frac{v^*}{U},\quad U(x)=\frac{U^*(x^*)}{U},\quad V(x)=\frac{V^*(x^*)}{U},\quad \beta=\frac{\lambda U}{L},\quad M=\frac{\sigma B_0^2 L}{\rho U},\quad \varepsilon=\frac{\mu}{\rho U L}, \tag{4}$$

the equations and boundary conditions are converted into

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{5}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\beta\left[u^2\left(\frac{\partial^2 u}{\partial x^2}\right)+v^2\left(\frac{\partial^2 u}{\partial y^2}\right)+2uv\left(\frac{\partial^2 u}{\partial x\partial y}\right)\right]=\varepsilon\left(\frac{\partial^2 u}{\partial y^2}\right)-M(x)u, \tag{6}$$

$$u(x,0)=U(x),\quad v(x,0)=V(x),\quad u(x,\infty)=0, \tag{7}$$

where β is the elasticity number, $M(x)$ the magnetic number, and $\varepsilon=1/\mathrm{Re}$. In the next section, symmetries of the above equations will be calculated assuming the physical parameters β and ε to be of order 1. If one or both of the parameters are small, an approximate symmetry theory may also be employed to enrich symmetries. See [30–32] for some of the different theories on approximate symmetries.

3. Lie Group Theory

Lie group theory is employed in search of symmetries of the equations. Details of the theory can be found in Bluman and Kumei [28] and Stephani [29]. The infinitesimal generator for the problem is

$$X=\xi_1(x,y,u,v)\frac{\partial}{\partial x}+\xi_2(x,y,u,v)\frac{\partial}{\partial y}+\eta_1(x,y,u,v)\frac{\partial}{\partial u}+\eta_2(x,y,u,v)\frac{\partial}{\partial v}. \tag{8}$$

Higher-order variables are defined next and the generator is prolonged to the higher-order variables. The infinitesimals of the higher-order variables are calculated using the standard recursion formulas given in [28, 29]. The prolonged generator is then applied to (5) and (6) yielding the invariance conditions which constitute a block of polynomial equations with respect to the higher-order variables. Separation with respect to higher-order variables and solving the simplest separated equations yield an over-determined linear system of equations (calculations verified by DOLIE symbolic program of MuMath developed by Alan Head [33]). Solving the over-determined linear system yields

$$\xi_1 = ax+b, \quad \xi_2 = cx+d, \quad \eta_1 = au, \quad \eta_2 = cu \tag{9}$$

with the below classifying relation

$$bM'(x) = 0, \quad aM'(x) = 0 \tag{10}$$

with respect to the variable magnetic number. Two cases arise.

3.1. Arbitrary $M(x)$

If M is an arbitrary function of the spatial variable x, then to satisfy (10), $a = 0$, $b = 0$. Hence the infinitesimals are

$$\xi_1 = 0, \quad \xi_2 = cx+d, \quad \eta_1 = 0, \quad \eta_2 = cu. \tag{11}$$

The above symmetries are the principal Lie algebra of the equations. For special forms of M, symmetries are expected to increase.

3.2. Constant M

From (10), the second alternative is $M'(x) = 0$, hence, M is either a constant or zero. Equations (10) are then satisfied and the infinitesimals are

$$\xi_1 = ax+b, \quad \xi_2 = cx+d, \quad \eta_1 = au, \quad \eta_2 = cu. \tag{12}$$

Results are summarized in Table 1.

Note that for constant M or non-existent magnetic field, there are four-parameter finite Lie group transformations whereas for arbitrary $M(x)$, they reduce to two-parameter finite Lie group transformations.

4. Restrictions Imposed by the Boundary Conditions

Usually boundary conditions put much restriction on the symmetries which may lead to a removal of all the symmetries. In our case, however, some of the symmetries remain stable after imposing the boundary conditions. For nonlinear equations, the generators should be applied to the boundaries and boundary conditions also [28]. Two cases are treated separately.

Table 1. Group Classification Summary.

Functional Dependence	Symmetries
M constant or zero	$\xi_1 = ax+b, \xi_2 = cx+d, \eta_1 = au, \eta_2 = cu$
$M(x)$ arbitrary	$\xi_1 = 0, \xi_2 = cx+d, \eta_1 = 0, \eta_2 = cu$

4.1. Arbitrary $M(x)$

For arbitrary $M(x)$ the generator was found to be

$$X = (cx+d)\frac{\partial}{\partial y} + cu\frac{\partial}{\partial v}. \tag{13}$$

Applying the generator to the boundary $y = 0$ yields $c = 0$ and $d = 0$, hence all symmetries are lost. One is left with $X = 0$, which means that for arbitrary $M(x)$, with the given boundary conditions, similarity solutions are impossible.

4.2. Constant M

For constant M, the generator was found to be

$$X = (ax+b)\frac{\partial}{\partial x} + (cx+d)\frac{\partial}{\partial y} + au\frac{\partial}{\partial u} + cu\frac{\partial}{\partial v}. \tag{14}$$

Application of the generator to the boundary $y = 0$ yields $c = 0$ and $d = 0$. Hence

$$X = (ax+b)\frac{\partial}{\partial x} + au\frac{\partial}{\partial u}. \tag{15}$$

Application of (15) to the boundary condition $u(x,0) = U(x)$ yields the differential equation

$$aU = (ax+b)U' \tag{16}$$

and application of (15) to $v(x,0) = V(x)$ yields

$$(ax+b)V' = 0. \tag{17}$$

The last condition $u(x,\infty) = 0$ does not impose further restrictions. If the injection velocity $V(x)$ is an arbitrary function and with the given boundary conditions $a = 0$ and $b = 0$ all symmetries are lost. Therefore the injection velocity can not be location dependent to ensure similarity transformations. It can be a constant or zero:

$$V(x) = v_0 \text{ or } V(x) = 0. \tag{18}$$

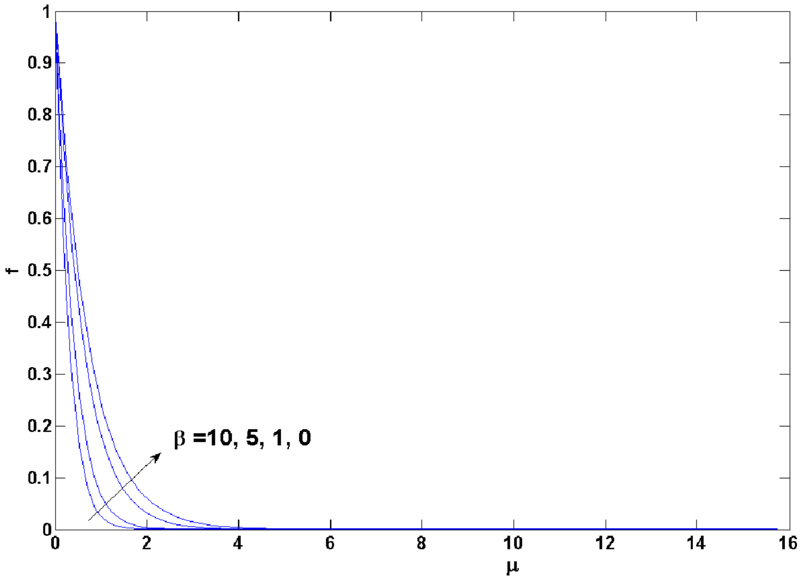

Fig. 1 (colour online). Effect of elasticity number on the similarity function f related to the x-component of velocity ($M = 1$, $\varepsilon = 1$).

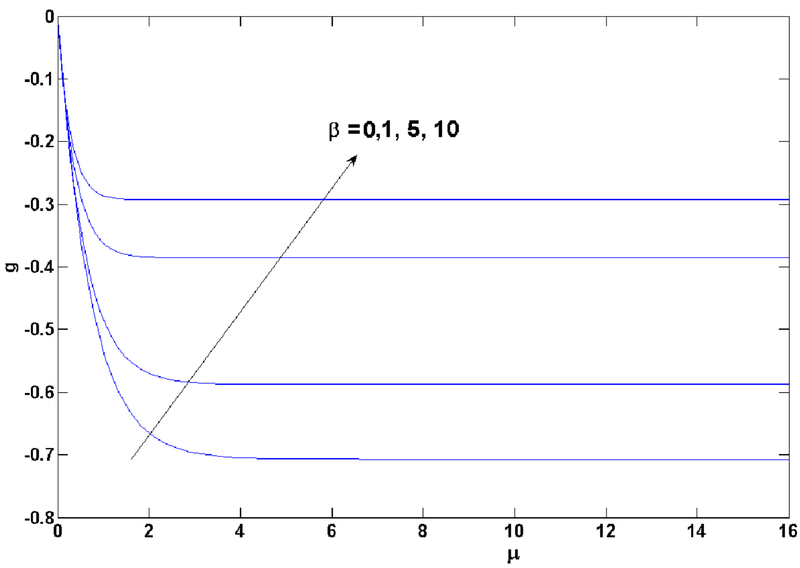

Fig. 2 (colour online). Effect of elasticity number on the similarity function g related to the y-component of velocity ($M = 1$, $\varepsilon = 1$).

If $U(x)$ is an arbitrary function of location, then again $a = 0$ and $b = 0$ and the symmetries are lost. However, for the specific form satisfying (16),

$$U(x) = u_0(ax + b), \tag{19}$$

a and b are nonzero and symmetries of the equation (generator (15)) are stable. This type of boundary condition corresponds to the stretching sheet case or the plate moving case with a constant velocity. In conclusion, if the injection velocity is a constant or zero and if the problem is a stretching sheet or constantly moving plate problem, similarity solutions exist for the constant magnetic number case. If $M(x)$ and $V(x)$ depend on x, no similarity solutions are possible. $U(x)$ on the other hand can utmost have a linear dependence on x as given in (19) to preserve symmetries.

5. Stretching Sheet Problem

In the light of the previous analysis, it is proven that similarity solutions exist for constant or zero magnetic field. It is also shown that the injection velocity should

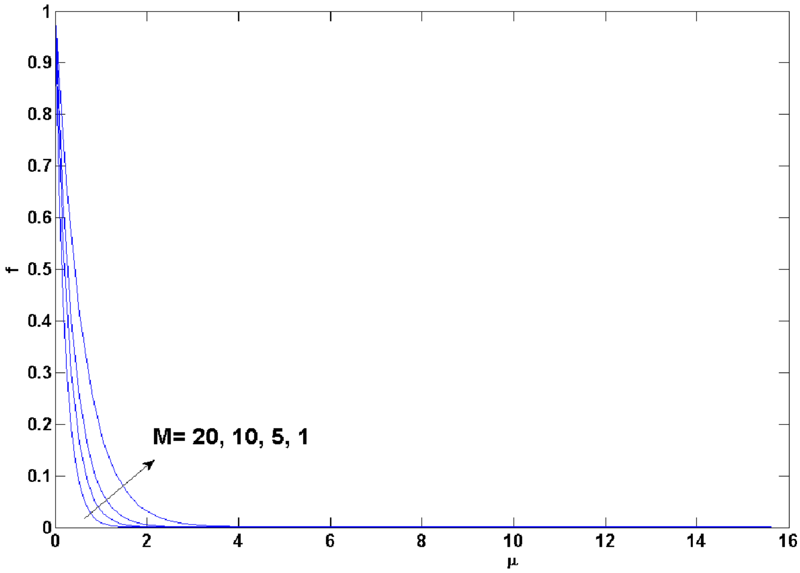

Fig. 3 (colour online). Effect of magnetic number on the similarity function f related to the x-component of velocity ($\beta = 1$, $\varepsilon = 1$).

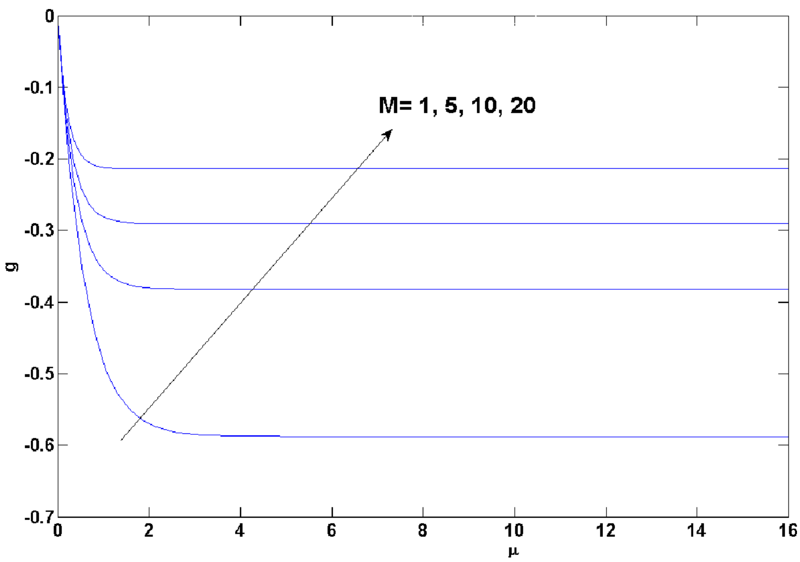

Fig. 4 (colour online). Effect of magnetic number on the similarity function g related to the y-component of velocity ($\beta = 1$, $\varepsilon = 1$).

be a constant or zero. The x velocity component can at most be a linear function in terms of the spatial variable. Therefore, the stretching sheet solution of a constant magnetic field case is treated in this section. Taking parameter 'a' in (15) and selecting $b = 0$, the associated equations which define the similarity variables are

$$\frac{\mathrm{d}x}{ax} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}u}{au} = \frac{\mathrm{d}v}{0}. \quad (20)$$

Solving the system yields the similarity variables

$$\mu = y, \quad u = xf(\mu), \quad v = g(\mu). \quad (21)$$

Substituting these into the boundary layer equations yields the ordinary differential system

$$f + g' = 0, \quad (22)$$

$$f^2 + gf' + \beta[g^2 f'' + 2ff'g] - \varepsilon f'' + Mf = 0. \quad (23)$$

The boundary conditions also transform to

$$f(0) = u_0, \quad g(0) = v_0, \quad f(\infty) = 0. \quad (24)$$

Without loss of generality u_0 is selected as 1. By using a special finite difference scheme, (22) and (23)

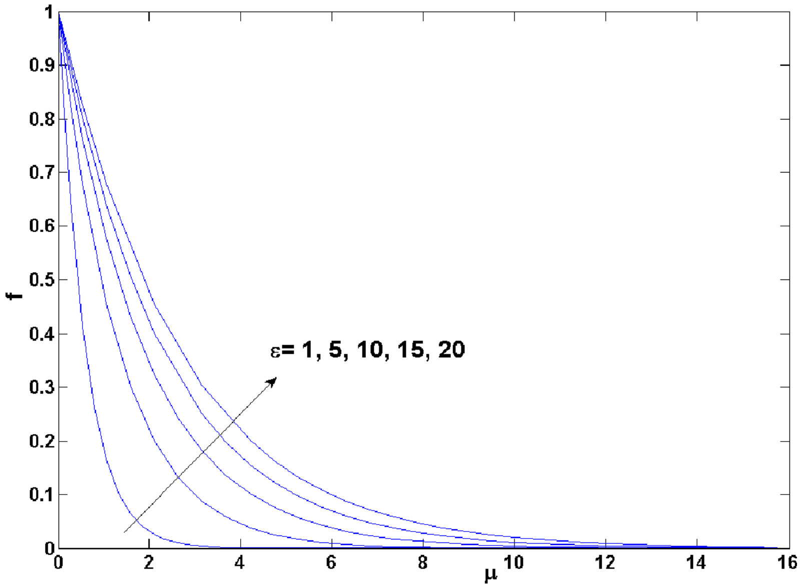

Fig. 5 (colour online). Effect of inverse Reynolds number on the similarity function f related to the x-component of velocity ($\beta = 1$, $M = 1$).

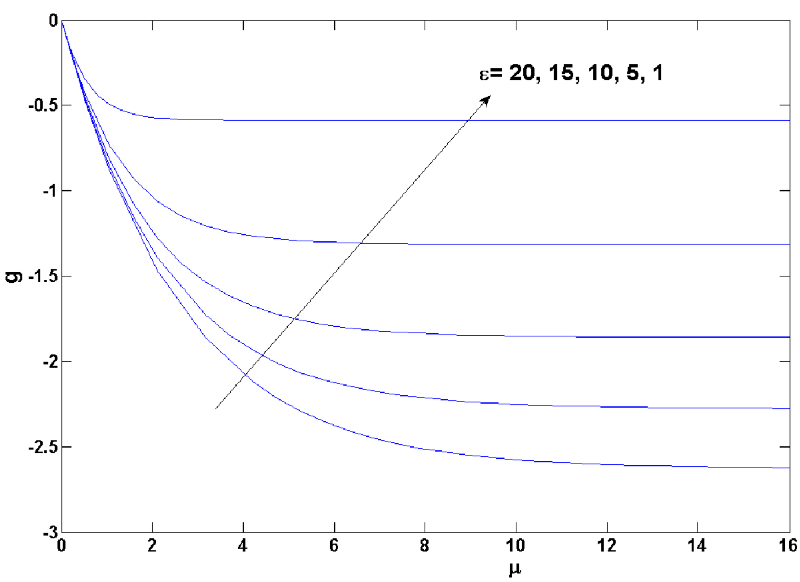

Fig. 6 (colour online). Effect of inverse Reynolds number on the similarity function g related to the y-component of velocity ($\beta = 1$, $M = 1$).

are integrated subject to the boundary conditions (24). In Figure 1, function f and in Figure 2 function g related to the x and y-components of the velocities are drawn for different elasticity numbers. With an increase in the elasticity number, the boundary layer becomes narrower. A decrease in f (related to the x-component of the velocity) and an increase in g (y-component of velocity) is observed. The effect of the magnetic field is depicted in Figures 3 and 4. As magnetic strength increases, the boundary layer narrows and an increase in the y-component of the velocity is observed. A converse effect is observed for the parameter $\varepsilon = 1/\mathrm{Re}$. ε is the ratio of the viscous forces to the inertia forces. As seen in Figures 5 and 6, the increase in ε thickens the boundary layer and lowers the y-component of the velocity. In Figures 1 – 6, there is no injection. Figures 7 and 8 compare the injection and the no-injection case. Injection thickens the boundary layer and increases the y-component of the velocity.

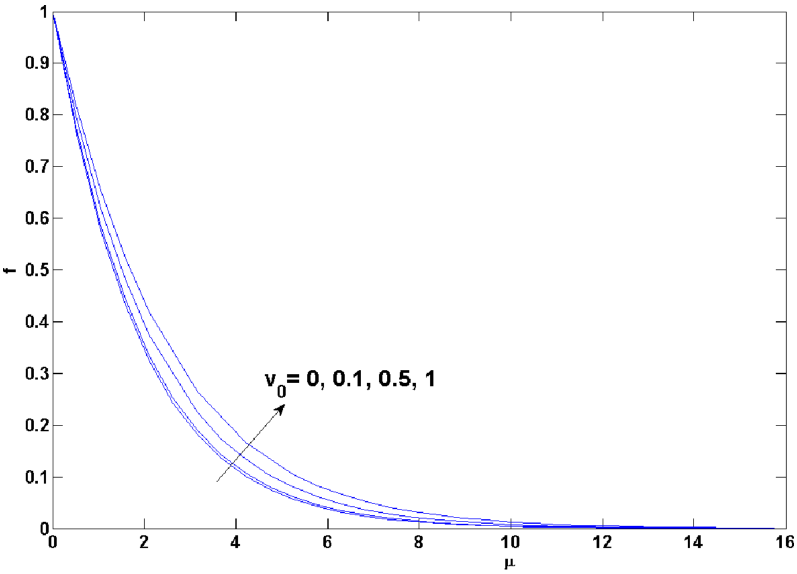

Fig. 7 (colour online). Effect of injection velocity on the similarity function f related to the x-component of velocity ($\beta = 1$, $M = 1$, $\varepsilon = 10$).

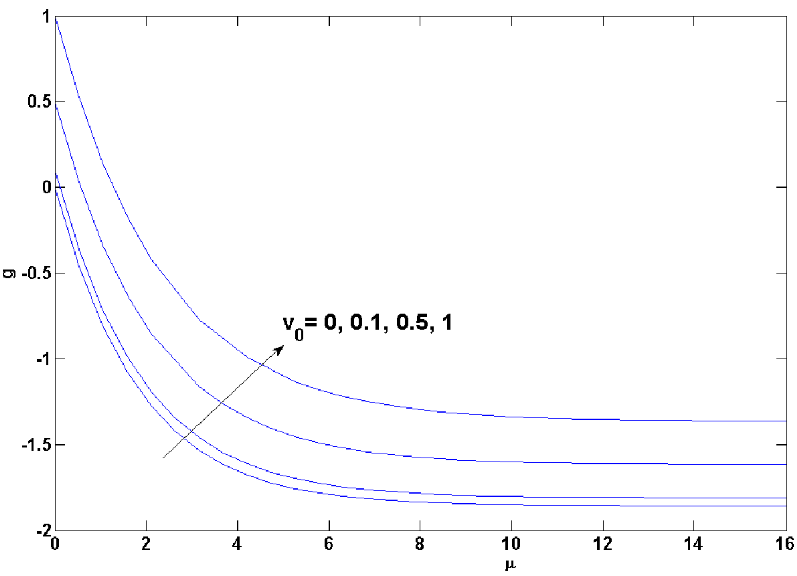

Fig. 8 (colour online). Effect of injection velocity on the similarity function g related to the y-component of velocity ($\beta = 1$, $M = 1$, $\varepsilon = 10$).

6. Concluding Remarks

Two-dimensional steady state boundary layer equations of an upper convected Maxwell fluid with variable magnetic field are considered. Lie group theory is applied to the equations. A group classification with respect to the variable magnetic field strength has been performed for the first time. It is found that the arbitrary magnetic field function case (principle Lie algebra) sustains two-parameter finite Lie group transformations whereas symmetries increase to four parameters for the constant or zero magnetic field case. Restrictions imposed by the boundary conditions on the symmetries are discussed. For a fairly general set of boundary conditions including stretching sheet, moving plate, and injection, the stability of the symmetries are discussed. For the variable magnetic field case, all symmetries are lost due to boundary conditions. For the constant or zero magnetic field case, the four-parameter Lie group transformations reduce to two parameters if the injection velocity is constant or zero and the velocity of the plate is a linear function of the spatial variable. From the remaining stable symmetries, a group invariant (similarity) solution is constructed. The partial differential equation system is converted to an ordinary differential system, and the resulting equations are solved by finite difference methods. Effects of elasticity number, magnetic number, inverse Reynolds number, and injection velocity on the solutions are outlined.

[1] A. Acrivos, M. J. Shahand, and E. E. Petersen, AIChE J. **6**, 312 (1960).
[2] W. R. Schowalter, AIChE J. **6**, 25 (1960).
[3] G. D. Bizzell and J. C. Slattery, Chem. Eng. Sci. **17**, 777 (1962).
[4] A. C. Srivastava, Z. Angew. Math. Phys. **9**, 80 (1958).
[5] D. W. Beard and K. Walters, Proceed. Cambridge Phil. Soc. **60**, 667 (1964).
[6] J. Astin, R. S. Jones, and P. Lockyer, J. Mech. **12**, 527 (1973).
[7] K. R. Rajagopal, A. S. Gupta, and A. S. Wineman, Int. J. Eng. Sci. **18**, 875 (1980).
[8] K. R. Rajagopal, A. S. Gupta, and T. Y. Na, Int. J. Nonlinear Mech. **18**, 313 (1983).
[9] V. K. Garg and K. R. Rajagopal, Acta Mechanica, **88**, 113 (1991).
[10] M. Massoudi and M. Ramezan, Int. J. Nonlinear Mech. **24**, 221 (1989).
[11] V. K. Garg and K. R. Rajagopal, Mech. Res. Commun. **17**, 415 (1990).
[12] M. Pakdemirli and E. S. Suhubi, Int. J. Eng. Sci. **30**, 523 (1992).
[13] M. Pakdemirli, Int. J. Nonlinear Mech. **27**, 785 (1992).
[14] M. Pakdemirli, IMA J. Appl. Math. **50**, 133 (1993).
[15] M. Pakdemirli, Int. J. Eng. Sci. **32**, 141 (1994).
[16] R. K. Bhatnagar, G. Gupta, and K. R. Rajagopal, Int. J. Nonlinear Mech. **30**, 391 (1995).
[17] T. Hagen and M. Renardy, J. Non-Newtonian Fluid Mech. **73**, 181 (1997).
[18] M. Renardy, J. Non-Newtonain Fluid Mech. **68**, 125 (1997).
[19] M. Yürüsoy and M. Pakdemirli, Int. J. Eng. Sci. **35**, 731 (1997).
[20] Y. Aksoy, M. Pakdemirli, and C. M. Khalique, Int. J. Eng. Sci. **45**, 829 (2007).
[21] K. Sadeghy, A. H. Najafi, and M. Saffaripour, Int. J. Nonlinear Mech. **40**, 1220 (2005).
[22] T. Hayat, Z. Abbas, and M. Sajid, Phys. Lett. A **358**, 396 (2006).
[23] Z. Abbas, M. Sajid, and T. Hayat, Theor. Comput. Fluid Dyn. **20**, 229 (2006).
[24] T. Hayat and Z. Abbas, Z. Angew. Math. Phys. **59**, 124 (2008).
[25] K. Sadeghy, H. Hajibeygi, and S. M. Taghavi, Int. J. Nonlinear Mech. **41**, 1242 (2006).
[26] A. A. Pahlavan and K. Sadeghy, Commun. Nonlinear Sci. Numer. Simul. **14**, 1355 (2009).
[27] V. Aliakbar, A. A. Pahlavan, and K. Sadeghy, Commun. Nonlinear Sci. Numer. Simul. **14**, 779 (2009).
[28] G. W. Bluman and S. Kumei, Symmetries and Differential Equations, Springer Verlag, New York 1989.
[29] H. Stephani, Differential Equations: Their Solution Using Symmetries, Cambridge University Press, New York 1989.
[30] A. H. Kara, F. M. Mahomed, and G. Ünal, Int. J. Theor. Phys. **38**, 2389 (1999).
[31] M. Pakdemirli, M. Yürüsoy, and Í. T. Dolapçı, Acta Appl. Math. **80**, 243 (2004).
[32] W. I. Fushchich and W. H. Shtelen, J. Phys. A **22**, L887 (1989).
[33] A. K. Head, Comput. Phys. Commun. **77**, 241 (1993).